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Mixed states in a neural network model

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Mixed states that are symmetrically related to more than one pattern are shown to exist in the Gardner model. Their retrievability and stability depend on the storage capacity of the network. We show that, under certain conditions, these states are stable and they have properties similar to those of the mixed states of the Hopfield model. [S1063-651X(98)50504-1]

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The Hopfield model [1,2] and the Gardner model [3] are widely used in neural networks. The synapses in the two models are generated differently. The synapses in the Hopfield model are determined by the Hebbian rule, while the synapses in the Gardner model are found by maximizing the storage capacity (see below) under the requirement of a desired input-output relation. Because of a number of appealing features such as flexibility in statistical analyses and large storage capacity, the Gardner model has become one of the important models. Both models have associative memory: the models are capable of retrieving the stored patterns by a stimulus of partial information about the patterns. The patterns in these models are stable states in the state space. In the Hopfield model there are additional states that are attractors but different from the stored patterns. Amit, Gutfreund, and Sompolinsky [2] have shown that there are metastable mixed states that are simultaneously related to more than one stored pattern. These metastable states are attractors and influence the dynamics of the network. For the Gardner model, one may ask: (1) Are there such mixed states? (2) If there are such states, are they stable? (3) How are these states related to the storage capacity of the model? The aim of this paper is to address these questions.

Since a full analytical treatment turns out to be impossible, we resort to numerical methods. However, the evolution through the first time step has been treated analytically by means of a dilution technique [4]. Our numerical and analytical results show that mixed states exist in the Gardner model and that they have properties similar to those of the Hopfield model [2].

In the Gardner model with N neurons, p patterns are remembered by the network when the condition

$$\frac{1}{\sqrt{N}}\xi_i^{\mu}\sum_{j\neq i}J_{ij}\xi_j^{\mu}>K\geqslant 0 \qquad (1)$$

is satisfied for all $i, i=1,\ldots,N$ and all μ, $\mu=1,\ldots,p$. A symmetrical mixed state has the same overlap (Hamming distance) with several patterns. In the Hopfield model, the mixed state can be obtained analytically from a mean-field equation. However, in the Gardner model, no analytical approach is known for obtaining such a state. Therefore we find the mixed state ζ from an initial state ξ' that is defined by the probability

$$p(\xi_i')=\frac{1}{n}\sum_{\mu=1}^{n}\delta(\xi_i'-\xi_i^{\mu}),\quad i=1,\ldots,N,1\leqslant n\leqslant p. \qquad (2)$$

Equation (2) means that ξ' is related to n patterns with the same probability $1/n$. The dynamics, when initiated with ξ', evolves to an equilibrium state that is the mixed state ζ for the Gardner model. In Eq. (2), $n=1$ corresponds to a nonmixed state, which is the usual case discussed in [3,5].

For studying the behaviors of states that lie in close proximity to ξ', we consider the time evolution of a typical state $S(t)$ that starts at t close to ξ'. Its evolution is studied by considering the Hamming distance (overlap) between one of the n patterns (e.g., ξ^1) and the state $S(t)$, $M(t)=(1/N)\Sigma_i^N\xi_i^1S_i(t)$. It may be pointed out that $M(t)=1/n$ when $S(t)=\xi'$. The overlap after one time step is obtained by generalizing the evaluation of [5,6]

$$M(t+1)=\int_{-\infty}^{+\infty}P_n(\Lambda)\,\mathrm{erf}\left(\frac{nM(t)\Lambda}{\sqrt{2\{1-[nM(t)]^2\}}}\right)d\Lambda \qquad (3)$$

with

$$P_n(\Lambda)=\left\langle\delta\left(\Lambda-\frac{1}{\sqrt{N}}\sum_{j\neq i}\xi_i^1J_{ij}\xi_j'\right)\right\rangle_{\xi',\xi}, \qquad (4)$$

where $\langle\cdots\rangle_{\xi',\xi}$ is the average over ξ' and ξ. After taking the average over ξ', the sum in Eq. (4) can be expressed as

$$\sum_{\mu=1}^{n}\frac{1}{\sqrt{N}}\xi_i^1\sum_{j\neq i}^{(\mu)}J_{ij}\xi_j^{\mu}, \qquad (5)$$

where, $\Sigma_{j\neq i}^{(\mu)}$ contains only the terms related to ξ^{μ} and totally it has N/n such terms. The sum $\Sigma_{j\neq i}^{(\mu)}$ can be understood as the corresponding quantity of a diluted network obtained by cutting randomly $N[1-1/(n)]$ synapses connecting to neuron i (in the fully connected network, the neuron i has $N-1$ synapses). Because $\xi_i^{\mu}=\pm 1$, ξ_i^1 in Eq. (5) can be replaced by ξ_i^{μ} with possible changes of sign, after considering all possible relations among $\xi_i^1,\xi_i^2,\ldots,\xi_i^n$. Hence, the sum in Eq. (5) has the form of $X_1\pm X_2\pm\cdots\pm X_n$ with

$$X_{\mu}=\frac{1}{\sqrt{N}}\xi_i^{\mu}\sum_{j\neq i}{}^{(\mu)}J_{ij}\xi_j^{\mu},\quad \mu=1,\ldots,n. \qquad (6)$$

The relation that brings l minus signs to the sum is described by a random walk procedure in which there are l steps walking toward the negative direction, after a movement of total $n-1$ steps. Equation (4) thus reads

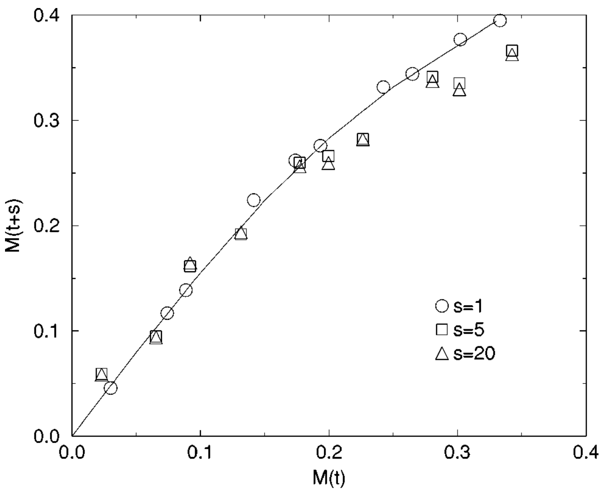

FIG. 1. The evolution of overlap $M(t)$ for $n=3$ at $K=2$. The solid line is the result of analytical treatment.

$$P_n(\Lambda)=\frac{1}{2^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}\langle\delta[\Lambda-f_l(X_1,X_2,\ldots,X_n)]\rangle_\xi, \tag{7}$$

where, $f_l(X_1,X_2,\ldots,X_n)$ is the algebraic sum $X_1\pm X_2\pm\ldots\pm X_n$ with l minus signs. The distribution $\langle\delta[\Lambda-f_l(X_1,X_2,\ldots,X_n)]\rangle_\xi$ can be obtained by

$$\langle\delta[\Lambda-f_l(X_1,X_2,\ldots,X_n)]\rangle_\xi = \int_{-\infty}^{+\infty}\left[\prod_{\mu=1}^{n}dX_\mu P_\mu(X_\mu)\right]\times\delta[\Lambda-f_l(X_1,X_2,\ldots,X_n)], \tag{8}$$

where $P_\mu(X_\mu)$ is just the distribution of X_μ in a quenched diluted network [4], where $N-(N/n)$ out of the total $N-1$ synapses have been removed randomly from every neuron.

Figure 1 gives the overlap $M(t+s)$ as a function of $M(t)$ after different time steps s for $n=3$. The solid line is the analytical result for $s=1$, while the circles, the squares and the triangles are the results of simulation after 1, 5, and 20 steps, respectively. The analytical and numerical overlaps after one step agree well. The overlap after one time step plays an important role in a saturated network [5]. Starting from ξ', after a certain number of time steps, the equilibrium state ζ is obtained when there is no more change in the overlap. Table I shows the overlap M_3 $[M_3=\lim_{s\to\infty}M(t+s)]$ of the mixed state for $n=3$ and various K. For $n=2,4,5,\ldots$, we obtained similar results.

In addition, we find an asymptotic maximum value 0.5 of M_3 for large K. In fact, such values (denoted by M_n^0) exist for all n. After some algebra to simplify the distribution of Eq. (8) for large K, M_n^0 is obtained from Eq. (3) as

$$M_n^0=\frac{1}{2^n}\binom{n}{\frac{n}{2}},\quad n=\text{ even} \tag{9}$$

and

$$M_n^0=\frac{1}{2^{n-1}}\binom{n-1}{\frac{n-1}{2}},\quad n=\text{ odd.} \tag{10}$$

These values coincide with those of the Hopfield model [2]. This provides evidence of similarity in structures of the mixed states of these two models.

Let us see now if the state is stable. The overlap, after long enough time, reaches an equilibrium value that alone does not provide information about its stability. Hence we turn to a suitable order parameter. According to the dynamics

$$S_i(t+1)=\text{ sgn}\left(\frac{1}{\sqrt{N}}\sum_{j\neq i}J_{ij}S_j(t)\right) =S_i(t)\text{ sgn}\left(\frac{1}{\sqrt{N}}\sum_{j\neq i}S_i(t)J_{ij}S_j(t)\right), \tag{11}$$

spin $S_i(t)$ flips if $\Sigma_{j\neq i}S_i(t)J_{ij}S_j(t)<0$, otherwise it stays unchanged. Therefore, the distribution $D(\Lambda)\equiv\langle\delta(\Lambda-\Sigma_{j\neq i}\zeta_iJ_{ij}\zeta_j/\sqrt{N})\rangle_\xi$ can be chosen as the parameter to judge the stability of ζ. We find that when $n=2$, D has always a finite value in the region of $\Lambda\leqslant 0$ for all K. Therefore, there are flips of a fraction of spins and this state ζ is not stable. Figure 2(a) gives the distribution $D(\Lambda)$ for $n=2$ at $K=1$ and 10. Even for very large K (e.g., $K=10$), the distribution is not zero for $\Lambda\leqslant 0$. This conclusion is also valid for even n. For $n=3$, we observe that the distribution is located at positive Λ, when K is large enough. Figure 2(b) shows the distribution for $n=3$. When $K=1.3$ (corresponding to the storage capacity $\alpha=0.4$), D exists also for $\Lambda\leqslant 0$; when $K=7$, the distribution resides only in the positive Λ region. The critical values of K (denoted by K_c) are determined numerically and are presented in Table II. Thus ζ is stable when $K>K_c$. This property can be generalized for all odd n, showing a similarity with the Hopfield model [2].

The different behaviors with even n and odd n may be explained by considering the distribution $P_n(\Lambda)$ of Eq. (7), which has maximal value at $\Lambda=0$ for even n when K is large. On the other hand, for odd n the maximum is reached at $\Lambda\neq 0$. With the dynamics, the distribution $P_n(\Lambda)$ evolves to $D(\Lambda)$ indicating a dynamical dependence of $D(\Lambda)$ on $P_n(\Lambda)$. From Eqs. (7),(8) and simulation results we observe that for even n there is a large number of spins (neurons) that maximize $P_n(\Lambda)$ near $\Lambda=0$. It is likely that this leads to nonzero $D(\Lambda)$ around $\Lambda=0$. In the case of odd n, however, there are fewer spins to provide nonzero $P_n(\Lambda)$ near $\Lambda=0$. This is perhaps the reason for zero $D(\Lambda)$ at $\Lambda=0$.

TABLE I. The overlap of ζ with pattern ξ^1 for $n=3$.

K	1	2	3	4	5	10
M_3	0.28	0.35	0.39	0.45	0.48	0.5

TABLE II. The critical values of K versus n.

n	3	5	7
K_c	5.5	8	10

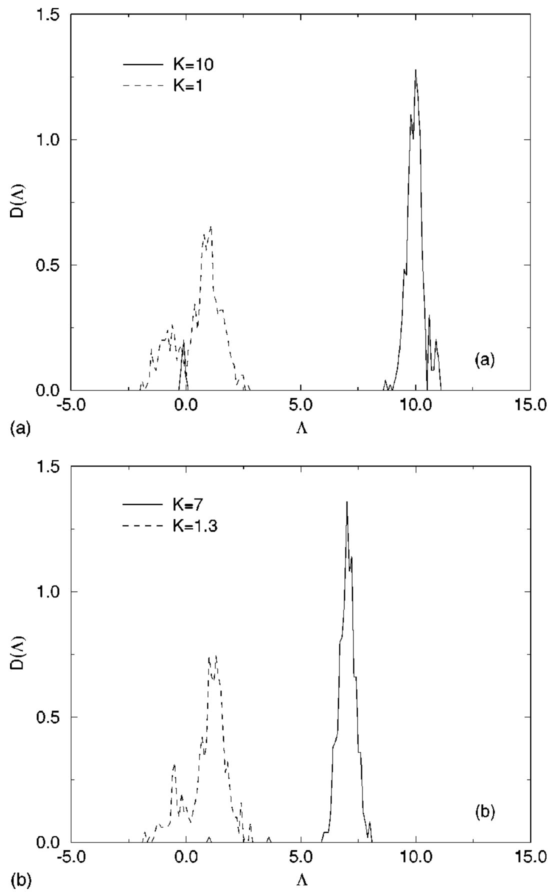

FIG. 2. The distribution D of state ζ for (a) $n=2$ at $K=1$ (dashed line) and $K=10$ (solid line); (b) $n=3$ at $K=1.3$ (dashed line) and $K=7$ (solid line).

The states ζ for odd n have a retrievability (attractor basin), when $K>K_c$. Their behavior can be shown clearly by introducing an overlap, $Q(t)=1/N\Sigma_i{}^N\zeta_i S_i(t)$, between ζ and a state S that is initially near ζ. Figure 3 gives the evolution of the overlap for $n=3$. After some steps ($s=20$ in Fig. 3) the overlap reaches an equilibrium value for $K>K_c$, while for $K<K_c$ there is no equilibrium value. In many cases of $K<K_c$, we observe two values appearing alternatively after one time step. That means there are two states, ζ_1 and ζ_2, which are reached from the same initial states with a cycle $\zeta_1\rightarrow\zeta_2\rightarrow\zeta_1\rightarrow\cdots$. We point out that both ζ_1 and ζ_2 have the same Hamming distance with any one of the n patterns. The cyclic states are therefore organized in an ultrametrical way [7]: they have the same macroscopical overlaps, but differ in the organization of their components. The states are separated by barriers that are related to the parameter K (the storage limit α). The two branches (dashed and long dashed lines in Fig. 3) of the overlap (corresponding to the two states) at smaller K tend to get closer to the solid curve at larger K. Therefore, for odd n, ζ has a basin of attraction depending on K as in the case of a single state and it forms a metastable state under the dynamics.

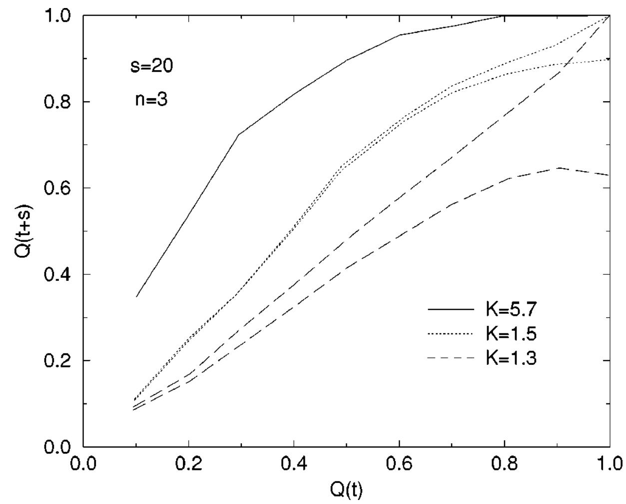

FIG. 3. The evolution of overlap $Q(t)$ for $n=3$ at $K=1.3$ (long dashed lines), $K=1.5$ (dashed lines), and $K=5.7$ (solid line).

The Edwards-Anderson parameter [8,9],

$$R=\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{i}^{N} S_i(t')S_i(t+t') \tag{12}$$

which compares the configurations at a different time, can also be introduced to study the stability of the states: R is 1 (<1), if the state is stable (unstable). Using R we have obtained the same results as discussed above.

In summary, we have shown both analytically and numerically that mixed states exist in the Gardner model of neural networks. We have also shown that the stability of these mixed states depends on the storage capacity of the model and also on whether n is even or odd. For every odd n, the capacity has a critical value below which the states are stable. In many respects, these states are similar to those of the Hopfield model.

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